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CENTRAL LIMIT ASYMPTOTICS FOR SHIFTS OF FINITE TYPE

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Dedicated to Horst Michel and his family

ABSTRACT

We study the rate of convergence and asymptotic expansions in the central limit theorem for the class of Hölder continuous functions on a shift of finite type endowed with a stationary equilibrium state. It is shown that the rate of convergence in the theorem is $O(n^{-1/2})$ and when the function defines a non-lattice distribution an asymptotic expansion to the order of $o(n^{-1/2})$ is given. Higher-order expansions can be obtained for a subclass of functions. We also make a remark on the central limit theorem for (closed) orbital measures.

0. Introduction

We consider the central limit theorem for a process $\{f \circ \sigma^n\}$ where σ is a shift of finite type endowed with a stationary equilibrium state m. We assume that f and the potential g defining m are Hölder continuous. For such a process we first prove the theorem with $O(n^{-1/2})$ rate of convergence and when f is not lattice distributed the rate is improved to $o(n^{-1/2})$. These results conform with the classical asymptotics of the central limit theorem for independent processes. However, we stress that the underlying stationary process we are considering has only a finite number of states which limits, for example, the entropy of the process $\{f \circ \sigma^n\}$. This limitation also inhibits us from imposing the classical condition used for higher asymptotics for independent processes. Nevertheless we find a reasonable substitute and

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exhibit a class of finite state independent processes satisfying this modified condition.

Related results of various strengths have previously appeared in connection with maps of the interval, with Axiom A diffeomorphisms, and indeed with shifts of finite type (cf. [DP], [Ke], [Ra], [Si], [Wo]). The paper [DP] is concerned with the approximation by Brownian motion of real-time processes which arise from Hölder suspensions of shift of finite type. The central limit theorem is obtained as a corollary but without significant asymptotics. Keller's work [Ke] on piecewise monotonic maps of the unit interval involves Hölder continuous functions together with Perron–Frobenius operators and the author is able to obtain the central limit theorem with $O(n^{-\theta})$ rate of convergence, where $\theta > 0$ is unspecified. The emphasis in our work, apart from an exceptionally simple presentation of the central limit theorem, is on the rate of convergence and asymptotic expansions. Under appropriate conditions we obtain an asymptotic expansion to the order of $o(n^{-1/2})$ and under further conditions this convergence rate is sharpened to give higher-order expansions. (However, see our remarks in the last section.) In view of Bowen's modelling theory for Axiom A diffeomorphisms by shifts of finite type our results, of course, apply to this context. We omit the details and refer the reader to Bowen's paper [Bo]. We conclude with remarks on the central limit theorem for (closed) orbital measures. This topic was recently considered by Lalley [La2].

The connection between the Ruelle operator and the central limit theorem was first indicated in an exercise in [Ru] — without asymptotics, however. Rousseau-Egèle [R-E] and Lalley [La1] also exploit this theme. This paper pursues a similar line of thought. The main ingredients of the method presented here are as follows:

(i) $\int \exp\{it(f + \cdots + f \circ \sigma^{n-1})n^{-1/2}\}dm$ can be written as

$$\int L^n_{g+itf/\sqrt{n}} 1 dm$$

where $L_{g+itf/\sqrt{n}}$ is a Ruelle operator which is a small perturbation of L_g;

(ii) if $\exp\{P(g + itf/\sqrt{n})\}$ is the maximum (simple, isolated) eigenvalue of $L_{g+itf/\sqrt{n}}$ then $\exp\{nP(g + itf/\sqrt{n})\}$ converges to $e^{-t^2\sigma^2/2}$, the Fourier transform of the normal distribution with variance σ^2.

Our basic theme is the exploitation of these observations. Readers are referred to [Ru] for the unproven statements in the next section.

1. Pressure and equilibrium states

Throughout we assume that A is a 0–1 $k \times k$ aperiodic matrix and we define

$$\Sigma_A^+ = \left\{ x \in \prod_{n=0}^{\infty} \{1, 2, \ldots, k\} : A(x_n, x_{n+1}) = 1, \text{ for all } n \in \mathbf{Z}^+ \right\}.$$

The space Σ_A^+ is a compact metrizable space with the Tychonov topology (generated by the discrete topology on $\{1, 2, \ldots, k\}$). The shift σ (of finite type) is defined on Σ_A^+ by $(\sigma x)_n = x_{n+1}$, $n \in \mathbf{Z}^+$, which is a continuous surjective map.

With $\mathrm{var}_n f = \sup\{|f(x) - f(y)| : x_i = y_i, i \leqq n\}$ and $0 < \theta < 1$, we define $|f|_\theta = \sup(\mathrm{var}_n f/\theta^n)$ and $F_\theta^+ = \{f \in C(\Sigma_A^+) : |f|_\theta < \infty\}$. F_θ^+ is a Banach space when endowed with the norm $\| f \|_\theta = |f|_\infty + |f|_\theta$, where $|\cdot|_\infty$ is the uniform norm.

Suppose that $f, g \in F_\theta^+$ are real valued functions and define the Ruelle operators L_{g+sf} on F_θ^+ by $(L_{g+sf}w)(x) = \Sigma_{\sigma y = x} \exp\{g(y) + sf(y)\}w(y)$. L_g has a maximum positive eigenvalue, denoted $e^{P(g)}$, which is simple and the rest of the spectrum is contained in a disc of radius strictly less than $e^{P(g)}$. $P(g)$ is called the *pressure* of g. There is a unique σ-invariant probability measure m such that $P(g) = h(m) + \int g dm$ which maximizes $h(\mu) + \int g d\mu$ for σ-invariant probabilities μ. The measure m is the *equilibrium state* of g. An eigenfunction w of L_g corresponding to $e^{P(g)}$ may be taken to be strictly positive and written $w = e^u$. If we replace g by $g' = g - P(g) + u - u \circ \sigma$ then we obtain $L_{g'}1 = 1$ and $P(g') = 0$. In this case we say that g' is normalised. It is easy to see that g and g' have the same equilibrium state m. *We shall always suppose that g is normalised and $\int f dm = 0$.*

Since the maximum eigenvalue 1 of L_g is isolated and simple, for $|s|$ sufficiently small each L_{g+sf} has a maximum eigenvalue (in modulus) $e^{P(g+sf)}$ which is also simple and the rest of the spectrum of L_{g+sf} is contained in a disc of radius strictly less than $|e^{P(g+sf)}|$. $P(g + sf)$ is an analytic function in, say, $|s| < \varepsilon$. In fact there exists an analytic projection valued function $Q(s)$ in $|s| < \varepsilon$, commuting with L_{g+sf}, so that $Q(s)1$ is "the" eigenvector of L_{g+sf} corresponding to $e^{P(g+sf)}$ satisfying:

(i) $|1 - e^{P(g+sf)}| < \eta$ and

(ii) L_{g+sf} restricted to $(\mathrm{Id} - Q(s))F_\theta^+$ has spectral radius less than ρ where $\rho < 1 - \eta$.

Let $w(s)$ denote $Q(s)1$. Differentiating the equation $L_{g+sf}w(s) = e^{P(g+sf)}w(s)$ with respect to s and using the fact that $L_{g+sf}(\cdot) = L_g(e^{sf}\cdot)$, we deduce that

$$L_{g+sf}(fw(s)+w'(s)) = e^{P(g+sf)}(P'(g+sf)w(s)+w'(s)) \tag{1}$$

and at $s=0$ we have

$$L_g(f+w'(0)) = P'(0)+w'(0). \tag{2}$$

Integrating with respect to m and using the fact that L_g^* fixes the measure m we obtain

$$P'(0) = \int f dm = 0. \tag{3}$$

A further differentiation of (1) yields

$$\begin{aligned} &L_{g+sf}(f^2w(s)+2fw'(s)+w''(s)) \\ &= e^{P(g+sf)}P'(g+sf)\{P'(g+sf)w(s)+w'(s)\} \\ &\quad + e^{P(g+sf)}\{P''(g+sf)w(s)+P'(g+sf)w'(s)+w''(s)\} \end{aligned} \tag{4}$$

and at $s=0$,

$$L_g(f^2+2fw'(0)+w''(0)) = e^{P(g)}(P''(0)+w''(0)). \tag{5}$$

Integrating (5) with respect to m we obtain

$$\int f^2 dm + 2\int fw'(0)dm = P''(0). \tag{6}$$

Applying the steps (1)–(6) to the equation $(L_{g+sf})^n w(s) = e^{nP(g+sf)}w(s)$ and noting that $(L_{g+sf})^n(\cdot) = (L_g)^n(e^{sf^n}\cdot)$ where $f^n(x) = f(x)+f(\sigma x)+\cdots+f(\sigma^{n-1}x)$ we get

$$\int (f^n)^2 dm + 2\int f^n w'(0)dm = nP''(0). \tag{7}$$

The Ergodic theorem shows that $(1/n)f^n \to 0$ m a.e., therefore we have

$$P''(0) = \lim_{n\to\infty} (1/n)\int (f^n)^2 dm. \tag{8}$$

This quantity will be denoted in the sequel by σ^2 *and we shall allways assume* $\sigma^2 \neq 0$. This condition is equivalent to the assumption that f *cannot* be written as $f = F\sigma - F$, with F continuous.

Differentiating once more we obtain the following expression for the third derivative of P:

$$P'''(0) = \lim_{n\to\infty} (1/n) \int (f^n - w'(0))^3 dm$$
$$= \lim_{n\to\infty} (1/n) \int ((f^n)^3 - 3(f^n)^2 w'(0)) dm.$$

For later reference we gather these observations:

LEMMA 1. *For $|s| < \varepsilon$, $P(g+sf)$ has an expression*

(9) $$P(g+sf) = \frac{\sigma^2 s^2}{2} + \frac{P'''(0)s^3}{6} + s^4 \varphi(s)$$

where $\varphi(s)$ is analytic.

2. Fourier transforms

In the whole of this section we will fix $\varepsilon > 0$ such that P is well-defined and analytic in $|s| < \varepsilon$; the projection $Q(s)$ of the last section is analytic in $|s| < \varepsilon$ and satisfies (i) and (ii); and ε satisfies

(10) $$\frac{\sigma^2}{2} > \varepsilon \left\{ \frac{|P'''(0)|}{6} + |s\varphi(s)| \right\} \quad \text{and} \quad \frac{\sigma^2}{2} > \varepsilon |\varphi(s)|$$

when $|s| < \varepsilon$. For the next result, let $\chi_n(t)$ denote $\int \exp\{itn^{-1/2} f^n\} dm$ where m is the equilibrium state of g with $g, f \in F_\theta^+$.

LEMMA 2. *If ε satisfies the conditions above then*

(11) $$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t} |\chi_n(t) - \exp\{nP(g + itf/\sqrt{n})\}| dt = O\left(\frac{1}{n}\right)$$

where the implied constant depends only on ε.

PROOF. We split the vector 1 analytically on the subspaces $Q(s)F_\theta^+$ and $(\mathrm{Id} - Q(s))F_\theta^+$, and since $1 - Q(s)1$ vanishes at $s = 0$, we obtain

(12) $$1 = w(s) + sv(s),$$

where $w(s) = Q(s)1$, for some analytic function $v(s)$ in $|s| < \varepsilon$. Since the vector $sv(s) \in (\mathrm{Id} - Q(s))F_\theta^+$, we conclude that $v(s)$ is fixed by the projection $\mathrm{Id} - Q(s)$ for $s \neq 0$.Therefore, by continuity, $(\mathrm{Id} - Q(0))v(0) = v(0)$. Integrating with respect to m we deduce that $\int v(0) dm = 0$. Hence

(13) $$\int v(s) dm = s\beta(s)$$

for some analytic function β in $|s| < \varepsilon$.

The integrand in (11) can be written

$$\frac{1}{t}\left|\int L^n_{g+itf/\sqrt{n}}1dm-\exp\{nP(g+itf/\sqrt{n})\}\right|$$

$$=\frac{1}{\sqrt{n}}\left|\int (L^n_{g+itf/\sqrt{n}}-\exp\{nP(g+itf/\sqrt{n})\})v(it/\sqrt{n})dm\right|.$$

The spectral radius of L_{g+sf} restricted to $(\mathrm{Id}-Q(s))F^+_\theta$ is less than ρ for each $|s|<\varepsilon$. From this we conclude that there exist K, ρ' ($\rho<\rho'<1$) such that

$$\| (L_{g+sf})^n v(s)\|_\theta \leqq K(\rho')^n \tag{14}$$

for all $|s|<\varepsilon$. In particular,

$$\left|\int (L_{g+sf})^n v(s)dm\right| \leqq K(\rho')^n,$$

and

$$\int_0^{\varepsilon\sqrt{n}}\left|\int L^n_{g+itf/\sqrt{n}}v(it/\sqrt{n})dm\right|dt \leqq K(\rho')^n\varepsilon\sqrt{n}. \tag{15}$$

Now from the choice of ε in (10) and the expression (9) of the pressure, we see that

$$|\exp\{nP(g+itf/\sqrt{n})\}| \leqq e^{-ct^2}$$

for a positive constant $c=c(\varepsilon)$ and all t, n satisfying $|t|\leqq\varepsilon\sqrt{n}$.

Since the function β defined in (13) is analytic in $|s|<\varepsilon$, it is clear that there exists a constant $K'=K'(\varepsilon)$ such that

$$\int_0^{\varepsilon\sqrt{n}} |\exp\{nP(g+itf/\sqrt{n})\}t\beta(it/\sqrt{n})|\,dt \leqq K'$$

for all $n>0$, and hence

$$\int_0^{\varepsilon\sqrt{n}}\left|\exp\{nP(g+itf/\sqrt{n})\}\left(\int v(it/\sqrt{n})dm\right)\right|dt \leqq K'/\sqrt{n}. \tag{16}$$

Combining (15) and (16) we have proved

$$\int_0^{\varepsilon\sqrt{n}}\left|\int (L^n_{g+itf/\sqrt{n}}-\exp\{nP(g+itf/\sqrt{n})\})v(it/\sqrt{n})dm\right|dt = O(1/\sqrt{n}) \tag{17}$$

where the implied constant depends only on ε, and the lemma follows.

THEOREM 1. *If ε satisfies the conditions above then*

(18) $$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t}\left|\chi_n(t) - e^{-\sigma^2 t^2/2}\left(1 - \frac{it^3P'''(0)}{6\sqrt{n}}\right)\right| dt = O\left(\frac{1}{n}\right)$$

where the implied constant depends only on ε.

PROOF. Using the elementary inequality

$$|e^{z+ib} - (1+ib)| \leqq |z|e^{|z|} + b^2/2$$

when b is real, we obtain from (9)

$$\exp\{nP(g + itf/\sqrt{n})\} = e^{-\sigma^2 t^2/2}\exp\left\{-\frac{it^3P'''(0)}{6\sqrt{n}} + \frac{t^4}{n}\,\varphi\left(\frac{it}{\sqrt{n}}\right)\right\}$$

$$= e^{-\sigma^2 t^2/2}e^{z+ib}$$

where we have defined $z = (t^4/n)\varphi(it/\sqrt{n})$ and $b = -t^3P'''(0)/6\sqrt{n}$.
Hence

(19) $$\begin{aligned}&|\exp\{nP(g+itf/\sqrt{n})\} - e^{-\sigma^2t^2/2}(1+ib)|\\ &\leqq e^{-\sigma^2t^2/2}|e^{z+ib} - (1+ib)|\\ &\leqq e^{-\sigma^2t^2/2}\left(\frac{t^4}{n}|\varphi|\exp\left\{\frac{t^4}{n}|\varphi|\right\} + \frac{t^6}{72n}|P'''(0)|^2\right).\end{aligned}$$

From the choice of ε in (10) we see that $\varepsilon|\varphi(it/\sqrt{n})| < \sigma^2/2$ for $t < \varepsilon\sqrt{n}$. Therefore (19) implies

$$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t}\left|\exp\{nP(g + itf/\sqrt{n})\} - e^{-\sigma^2t^2/2}\left(1 - \frac{it^3P'''(0)}{6\sqrt{n}}\right)\right| dt \leqq \frac{K}{n}$$

for some K depending only on ε. Combining this inequality with Lemma 2 we have proved the theorem.

3. Refinements in the central limit theorem

Consider $f, g \in F_\theta^+$ and m the equilibrium state of g. We assume $\int f dm = 0$ and consider the distribution function $F_n(x) = m\{y \in \Sigma_A^+ : n^{-1/2}f^n(y) < x\}$. N denotes the normal distribution with zero mean and variance $\sigma^2 \neq 0$. The central limit theorem states that $|F_n(x) - N(x)| = o(1)$, uniformly in x. We are interested in the asymptotic behaviour of this convergence.

The function $\chi_n(t)$ given by $\int \exp\{itn^{-1/2}f^n\}dm$ is the Fourier transform of

$F_n(x)$. If $\gamma(t)$ is the Fourier transform of the distribution $G(x)$ then a well-known "basic inequality" (cf. [Fe]) asserts that

$$\| F_n(x) - G(x) \| \leqq \frac{1}{2\pi} \int_0^T \frac{1}{t} |\chi_n(t) - \gamma(t)| \, dt + \frac{24M}{\pi T} \tag{20}$$

where M is the maximum value of the derivative G' of G and T is arbitrary.

THEOREM 2. $\| F_n(x) - N(x) \| = O(1/\sqrt{n})$.

PROOF. Recall that $e^{-\sigma^2 t^2/2}$ is the Fourier transform of $N(x)$. Taking G as the normal distribution and setting $T = \varepsilon n^{1/2}$ in (20), where ε satisfies the conditions in (10), by applying Theorem 1 of the last section we obtain the result.

We say that f defines a *non-lattice distribution* when, for all $a \in \mathbf{R}$, the values $f^n(x) + na$ for points x of period n generate a dense subgroup of $\mathbf{R}$. The assumption that f defines a non-lattice distribution is equivalent to the condition that whenever $\varphi \circ \sigma = \alpha \exp\{itf\}\varphi$ (for a continuous, or even measurable φ) then $t = 0$ and φ is constant. Alternatively, the condition is equivalent to the assumption that L_{g+itf} has spectral radius strictly less than 1, when $t \neq 0$ (cf. [Po1]). In the special case when $\{f \circ \sigma^n\}$ is an independent sequence, the above concept is implied by the standard notion which is defined, for example, in [Fe].

THEOREM 3. *If f defines a non-lattice distribution then*

$$F_n(x) - N(x) = \frac{P'''(0)}{6\sigma^3\sqrt{2\pi n}} \left(1 - \frac{x^2}{\sigma^2}\right) e^{-x^2/2\sigma^2} + o\left(\frac{1}{\sqrt{n}}\right).$$

PROOF. Define

$$G_n(x) = N(x) + \frac{P'''(0)}{6\sigma^3\sqrt{2\pi n}} \left(1 - \frac{x^2}{\sigma^2}\right) e^{-x^2/2\sigma^2}.$$

Its Fourier transform is given by

$$\gamma_n(t) = e^{-\sigma^2 t^2/2}\left(1 + \frac{P'''(0)(it)^3}{6\sqrt{n}}\right).$$

If we prove that for any fixed $\varepsilon > 0$ and $\alpha > \varepsilon$

$$\int_{\varepsilon\sqrt{n}}^{\alpha\sqrt{n}} \frac{1}{t}(\chi_n(t) - \gamma_n(t))dt \to 0$$

exponentially fast as $n \to \infty$, then considering the basic inequality (20) with $T = \alpha n^{1/2}$, $G(x) = N(x)$ and $F_n(x)$ replaced by $F_n(x) - G_n(x) + N(x)$, and using the fact that the second term of the right-hand side of (20) can be made smaller than any pre-assigned quantity (by choosing a large α), we can apply Theorem 1 to establish the result.

Hence it remains to prove that

$$\int_{\varepsilon\sqrt{n}}^{\alpha\sqrt{n}} \frac{1}{t} |\chi_n(t)| dt \to 0$$

at an exponential rate for any fixed pair ε, α strictly positive. The latter integral is equal to (after a change of variables $y = tn^{-1/2}$)

$$\int_\varepsilon^\alpha \left| \int \exp\{iyf^n\} dm \right| \frac{dy}{y} = \int_\varepsilon^\alpha \left| \int L_{g+iyf}^n 1 dm \right| \frac{dy}{y}.$$

Since the function f defines a non-lattice distribution, the operator L_{g+iyf} has spectral radius strictly less than 1 for each real $y \neq 0$, therefore the latter integral converges to zero exponentially fast as $n \to \infty$, since y is bounded away from zero.

REMARK. The above theorem remains valid if we consider a two-sided shift in place of the one-sided shift we have been considering. This is because the function f may be replaced by $f' = f + F\sigma - F$ with F continuous, without altering the asymptotic. However, we see no reason which would justify a similar claim for higher-order asymptotic assertions.

4. Higher-order asymptotics

In this section we indicate how the previous results may be improved to an $o(n^{-r/2+1})$ asymptotic. Since the method does not involve anything new, we shall concentrate on one or two of the main features.

For $r \geqq 3$ we define a polynomial $\psi_r(s)$ of degree $r-2$, by

$$P(g + sf) - \frac{s^2\sigma^2}{2} = s^2\psi_r(s) + o(s^r)$$

where $\psi_r(0) = 0$.

Then

$$\left| \exp\left\{ nP(g+sf) - \frac{ns^2\sigma^2}{2} \right\} - \sum_{k=0}^{r-2} \frac{(ns^2\psi_r(s))^k}{k!} \right|$$

$$\leqq no(s^r)e^\gamma + \left| \frac{ns^2\psi_r(s)}{(r-1)!} \right|^{r-1}$$

where $\gamma \leqq n|s|^3K$ for some $K = K(\varepsilon)$ and $|s| < \varepsilon$, when $\varepsilon > 0$ is suitably small.

With $s = it/\sqrt{n}$ ($|t| < \varepsilon\sqrt{n}$) we have

$$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t} \left| \exp\{nP(g + itf/\sqrt{n})\} - e^{-\sigma^2 t^2/2} \sum_{k=0}^{r-2} \frac{(-t^2\psi_r(it/\sqrt{n}))^k}{k!} \right| dt \tag{21}$$

$$\leqq \frac{\delta A}{n^{r/2-1}} + \frac{B}{n^{r/2-1/2}}$$

where A, B are absolute constants (depending on r) and $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

As before the upper limit of the integral in (21) may be increased without harm to $a\sqrt{n}$ for any positive a. But to apply the basic inequality and obtain an $o(n^{-r/2+1})$ asymptotic we have to ensure, say, that

$$\int_{an^{1/2}}^{an^{r/2-1}} \left(\int \exp\{itf^n/\sqrt{n}\} dm \right) \frac{dt}{t} \tag{22}$$

converges to zero rapidly.

A standard condition imposed when $\{f \circ \sigma^n\}$ is an independent process is that $\limsup |\chi(t)| < 1$ where $\chi(t) = \int e^{itf} dm$, for this implies that

$$\int e^{itf^n} dm \to 0$$

exponentially fast, uniformly for large t. However, this is an inappropriate condition for our situation. (Indeed, it may never be satisfied when f depends on only one variable.)

The following condition is sufficient:

$$\mathrm{H}_r: \qquad \left| \int e^{itf^n} dm \right| \leqq K(1 - c/t^\alpha)^n$$

for constants c, K, α with $\alpha(r-3)/2 < 1$ and all sufficiently large t.

With this condition we have

$$\left| \int_{an^{1/2}}^{an^{r/2-1}} \left(\int \exp\{itf^n/\sqrt{n}\}dm \right) \frac{dt}{t} \right|$$

$$= \left| \int_{a}^{an^{(r-3)/2}} \left(\int \exp\{itf^n\}dm \right) \frac{dt}{t} \right|$$

$$\leqq (n^{(r-3)/2} - 1)K\left(1 - \frac{c}{n^{\alpha(r-3)/2}}\right)^n$$

which tends to zero faster than the reciprocal of any polynomial.

We therefore have

THEOREM 4. *If condition H_r is satisfied ($r \geqq 3$) then there exist polynomials R_k depending only on the first r derivatives of $P(g + sf)$ at $s = 0$ and on the first $r - 2$ derivatives of $\int v(s)dm$ at $s = 0$ such that*

$$F_n(x) - N(x) = e^{-x^2/2\sigma^2} \sum_{k=3}^{r} n^{-k/2+1}R_k(x) + o(n^{-r/2+1}).$$

5. Arithmetical Bernoulli processes

Here we wish to show that the conditions H_r are non-empty by constructing certain independent processes where $f(x) = f(x_0)$ has its range with specific arithmetical properties. These examples were previously studied by Pollicott [Po2] in connection with the extension domain of meromorphy of the dynamical zeta function.

First we note the following elementary

LEMMA 3. *Let $w_0, \ldots, w_d$ be complex numbers of modulus 1. Let $(p_0, \ldots, p_d)$ be a probability vector with $p_j > 0$ for all j. Then*

$$\left| \sum_{i=0}^{d} p_i w_i \right| \leqq 1 - \frac{\mu^2}{4\left(\frac{1}{p_i} + \frac{1}{p_j}\right)}$$

where $\mu = |w_i - w_j|$ is the maximum difference.

PROOF. Let $\mu = |w_i - w_j|$ be the maximum difference. Clearly

$$\left| \sum_{k=0}^{d} p_k w_k \right| \leqq 1 - (p_i + p_j) + |p_i w_i + p_j w_j|.$$

Hence it suffices to show

$$\left| \frac{p_i w_i + p_j w_j}{p_i + p_j} \right| \leqq 1 - \frac{\mu^2 p_i p_j}{4(p_i + p_j)^2},$$

which follows from the lemma for $d = 1$. Therefore it remains to prove that given a, p, q ($p, q > 0$, $p + q = 1$) then

$$| p + qe^{ia} | \leqq 1 - \frac{|1 - e^{ia}|^2 pq}{4},$$

which is then a straightforward computation.

Lemma 4. *With $w_j = e^{ia_j t}$ where, say, $a_1 - a_0, \ldots, a_d - a_0$ are algebraic numbers which are rationally independent (this condition is invariant under permutations), for each $\delta > 0$, we have*

$$\left| \sum_{j=0}^{d} p_j e^{ia_j t} \right| \leqq 1 - \frac{c}{t^{\{2/(d-1)\}+2\delta}}$$

for some constant $c > 0$ and all $t > t_0$ (t_0 depending on δ), where $(p_0, \ldots, p_d)$ is a probability vector with $p_j > 0$ for all j.

Proof. Since $a_1 - a_0, \ldots, a_d - a_0$ are rationally independent, so are

$$1, \frac{a_2 - a_0}{a_1 - a_0}, \ldots, \frac{a_d - a_0}{a_1 - a_0}.$$

By the celebrated generalisation of the Thue–Siegel–Roth theorem due to W. Schmidt [Sc] there exist c' and $n_0 = n_0(\delta)$ such that for all $n > n_0$ at least one of the inequalities

$$\left\| n \left(\frac{a_j - a_0}{a_1 - a_0} \right) \right\| \geqq \frac{c'}{n^{\{1/(d-1)\}+\delta}}$$

holds ($j = 2, \ldots, d$). Here $\| x \| = \inf\{ |x + m| : m \in \mathbf{Z}\}$.

From this we see that

$$\max_{1 \leqq j \leqq d} | e^{2\pi i t(a_j - a_0)} - 1 | \geqq \frac{c}{t^{\{1/(d-1)\}+\delta}}$$

for all $t > t_0$ (and some constant $c > 0$). Hence by the previous lemma

$$\left| \sum_{i=0}^{d} p_j e^{ia_j t} \right| \leqq 1 - \frac{c^2 p}{8t^{\{2/(d-1)\}+2\delta}}$$

where p is the least p_j.

Now suppose that f' is defined on the full $d+1$ shift endowed with the Bernoulli measure m generated by $(p_0, \ldots, p_d)$. We suppose f' is a function of one variable, in fact $f'(j) = a_j$ where $a_1 - a_0, \ldots, a_d - a_0$ are rationally independent algebraic numbers. Finally we define $f = f' - \int f' dm$. Clearly f enjoys the same arithmetical properties. Hence, using Lemma 4,

$$\left| \int e^{itf^n} dm \right| = \left| \int e^{itf} dm \right|^n$$

$$\leqq \left(1 - \frac{c}{t^{2/(d-1)+2\delta}}\right)^n.$$

Thus if $(r-3)/(d-1) < 1$ then f satisfies H_r.

6. Remark on orbital measures

As a final remark we consider the central limit theorem for $f \in F_\theta^+$ ($\int f dm = 0$) with the equilibrium state m replaced by the sequence of orbital measures m_n:

$$\int k dm_n = \frac{\sum_{\mathrm{Fix}(n)} k(x) \exp\{g^n(x)\}}{\sum_{\mathrm{Fix}(n)} \exp\{g^n(x)\}}$$

where $\mathrm{Fix}(n) = \{x : \sigma^n(x) = x\}$.

For a proof that $m_n \to m$ weakly, cf. [Ru].

The central limit theorem for this sequence states:

THEOREM 5. *$m_n\{x : f^n(x)/\sqrt{n} < y\} \to N(y)$, as $n \to \infty$.*

To see that the refinements of earlier sections apply to this theorem also, we note the following lemma (cf. [Ru]):

LEMMA 5. *There exists $\varepsilon > 0$ such that for $|s| < \varepsilon$,*

$$\limsup_n \left| \sum_{\mathrm{Fix}(n)} e^{g^n(x) + sf^n(x)} - e^{nP(g+sf)} \right|^{1/n} < 1.$$

As a consequence we have

$$\left| \int \exp\{itf^n/\sqrt{n}\}dm_n - \exp\{nP(g+itf/\sqrt{n})\} \right|$$

$$= \left| \frac{\sum_{\mathrm{Fix}(n)} e^{g^n(x)+itf^n(x)/\sqrt{n}}}{\sum_{\mathrm{Fix}(n)} e^{g^n(x)}} - \exp\{nP(g+itf/\sqrt{n})\} \right| \leqq K\rho^n$$

for constants $K = K(\varepsilon)$, ρ $(0 < \rho < 1)$ *for all* $|t| < \varepsilon\sqrt{n}$.

7. Conclusion

The referee has kindly raised two problems which follow from Theorems 1 and 4. At the present time we are unable to solve them but it seems to us that they are well worth proposing as open problems:

(i) Theorem 1 is a version of the Berry–Essen theorem (cf. [Fe]) but without the implied constant being specified. Can a reasonable constant be found, depending on the first few derivatives of the pressure function, and perhaps the first few derivatives of $w(s)$?

(ii) How restrictive is the condition H_r in Theorem 4? The same question, of course, applies to any reasonable substitute for H_r.

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